

CREPANT RESOLUTIONS, QUIVERS AND GW/NCDT DUALITY

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ABSTRACT. We propose a conjecture that relates some local Gromov-Witten invariants of some crepant resolutions of Calabi-Yau 3-folds with isolated singularities with some Donaldson-Thomas type invariants of the moduli spaces of representations of some quivers with potentials.

1. INTRODUCTION

Singular Calabi-Yau 3-folds and their crepant resolutions are very interesting both in algebraic geometry and in string theory. Not only do they provide numerous interesting examples of resolutions of singularities, but also the study of their invariants is very rich. Given a crepant resolution $\pi : Y \rightarrow X$ of an affine Calabi-Yau 3-fold with an isolated singularity, the local Gromov-Witten invariants and the derived category of coherent sheaves of Y are of particular interest. They can be studied both from the algebrogeometric point of view and the string theoretical point of view, and the interactions are very crucial for some of the recent progresses. Gromov-Witten invariants are defined mathematically as intersection numbers on the moduli spaces of stable maps to Y , they correspond to some correlators in type IIA closed string theory; objects in derived category of coherent sheaves are complexes of coherent sheaves up to some equivalence relations, they correspond to type IIB D-branes. Let us recall how D-branes lead us to quivers with superpotentials functions. Physically Type B D-brane at a singular point can decay into a collection of stable fractional D-branes. Each constituent D-brane may appear with a multiplicity, and there are open strings between them. In this way one can associate a quiver gauge theory to it. Mathematically, this is described by a noncommutative resolution [35]: one can often find a quiver with superpotential (Q, W) such that there is an equivalence:

$$D^b(Y) \cong D^b(Q, W),$$

where $D^b(Y)$ is the derived category of coherent sheaves on Y , and $D^b(Q, W)$ is the derived category of quiver representations of Q constrained by the relations given by the superpotential function. The path algebra of the quiver Q with relations given by W is noncommutative, called a Calabi-Yau algebra [6, 14]. One can define Donaldson-Thomas type invariants of the moduli spaces of semistable representations of (Q, W) . We conjecture that they are related to the Gromov-Witten invariants of Y . For more precise formulation, See Conjecture 1 in Section 3.

The rest of the paper is organized as follows. In Section 2 we will recall some examples of crepant resolutions of affine Calabi-Yau 3-folds with isolated singularities and the quivers with superpotentials associated to them. In Section 3 we will present our conjecture.

2. CREPANT RESOLUTIONS OF CALABI-YAU 3-FOLDS AND QUIVERS WITH POTENTIALS

In this section we will recall some well-known examples of crepant resolutions and their associated quivers with superpotentials.

2.1. The case of \mathbb{P}^1 as exceptional sets. By a result due to Laufer [23], when the exceptional set is \mathbb{P}^1 , its normal bundle is isomorphic to one of the following three bundles: $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\mathcal{O}(-2) \oplus \mathcal{O}$, and $\mathcal{O}(-3) \oplus \mathcal{O}(1)$. The first case can be realized by the well-known resolved conifold, and the their local Gromov-Witten invariants are well-known [17, 12]; the other two case are realized by Laufer's examples [23], for their local Gromov-Witten invariants, and more generally, that of $\mathcal{O}(k) \oplus \mathcal{O}(-k-2) \rightarrow \mathbb{P}^1$ for $k \geq -1$ see [36].

The resolved conifold can be obtained by gluing two copies of \mathbb{C}^3 , with linear coordinates (x, y_1, y_2) and (w, z_1, z_2) respectively, by the following formula for change of coordinates:

$$z_1 = xy_1, \quad z_2 = xy_2, \quad w = \frac{1}{x}.$$

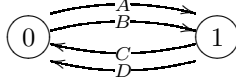
Let

$$\begin{aligned} v_1 &= z_1 = xy_1, & v_2 &= z_2 = xy_2, \\ v_3 &= wz_1 = y_1, & v_4 &= wz_2 = y_2. \end{aligned}$$

They satisfy:

$$(1) \quad v_1 v_4 - v_2 v_3 = 0.$$

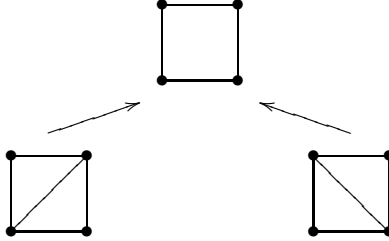
Hence we get a contraction $\pi : Y \rightarrow X := \{(v_1, v_2, v_3, v_4) \in \mathbb{C}^4 : v_1 v_4 - v_2 v_3 = 0\}$. The associated quiver is (cf. [3] and the references therein):



with superpotential function:

$$W = BCAD - ACBD.$$

This case has been studied by Nagao and Nakajima [28]. Note the conifold and the resolved conifold are toric Calabi-Yau 3-folds. In general we will consider toric affine Calabi-Yau 3-folds that correspond to the polygons with integral vertices in \mathbb{Z}^2 . A toric crepant resolution corresponds to a subdivision into triangles with integral vertices and of area $1/2$. If two subdivisions differ only by changing the diagonal of a parallelogram, then the corresponding crepant resolutions differ by a flop. For example, the conifold corresponds to the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. It has two subdivisions:



Now we come to Laufer's examples. The first class of examples is the space Y obtained by gluing two copies of \mathbb{C}^3 , with linear coordinates (x, y_1, y_2) and (w, z_1, z_2) respectively, by the following rules for change of coordinates:

$$z_1 = x^2 y_1 + x y_2^k, \quad z_2 = y_2, \quad w = \frac{1}{x}.$$

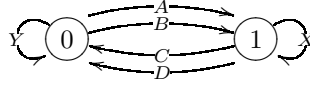
Define a map from Y to \mathbb{C}^4 by:

$$\begin{aligned} v_1 &= z_2 = y_2, & v_2 &= z_1 = x^2 y_1 + x y_2^k, \\ v_3 &= w z_1 = x y_1 + y_2^k, & v_4 &= w^2 - w z_2^k = y_1. \end{aligned}$$

The image is the affine variety X defined by $v_2 v_4 - v_3^2 + v_3 v_1^k = 0$, or by a change of coordinates:

$$(2) \quad u_1^2 + u_2^2 + u_3^2 + u_4^{2k} = 0.$$

The exceptional set is given by $(x, y_1 = 0, y_2)$ and $(w, z_1 = 0, z_2 = 0)$. Its normal bundle is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ when $k = 1$, and to $\mathcal{O} \oplus \mathcal{O}(-2) \rightarrow \mathbb{P}^1$ when $k > 1$. For $k > 1$, the associated quiver is (cf. [3] and the references therein):



with superpotential function

$$W = -(-1)^{n(n-1)/2} X^{n+1} - (-1)^{n(n-1)/2} Y^{n+1} - XAC + XBD - YCA + YDB.$$

These examples are not toric, nevertheless there is a natural 2-torus action defined as follows:

$$\begin{aligned} (t_1, t_2) \cdot (x, y_1, y_2) &\mapsto (t_1^{-1} t_2^k x, t_1 y_1, t_2 y_2), \\ (t_1, t_2) \cdot (w, z_1, z_2) &\mapsto (t_1 t_2^{-k} w, t_1^{-1} t_2^{2k} z_1, t_2 z_2), \end{aligned}$$

where $t_1, t_2 \in \mathbb{C}^*$. Using this action one can define and compute the local Gromov-Witten invariants as in [36].

The second class of examples is the space Y obtained by gluing two copies of \mathbb{C}^3 , with linear coordinates (x, y_1, y_2) and (w, z_1, z_2) respectively, by the following formula for change of coordinates:

$$z_1 = x^3 y_1 + y_2^2 + x^2 y_2^{2n+1}, \quad z_2 = \frac{y_2}{x}, \quad w = \frac{1}{x}.$$

Define a map $Y \rightarrow \mathbb{C}^4$ by:

$$\begin{aligned} v_1 &= z_1 & &= x^3 y_1 + x y_2^{2n+1}, \\ v_2 &= w^2 z_1 - z_2^2 & &= x y_1 + y_2^{2n+1}, \\ v_3 &= w^3 z_1 - w z_2^2 - z_1^n z_2 & &= y_1 + \frac{1}{x} [y_2^{2n+1} - y_2 (x^3 y_1 + y_2^2 + x^2 y_2^{2n+1})^n], \\ v_4 &= w^2 z_1 z_2 - z_2^3 - w z_2^{n+1} & &= y_1 y_2 + \frac{1}{x} [y_2^{2n+2} - (x^3 y_1 + y_2^2 + x^2 y_2^{2n+1})^{n+1}]. \end{aligned}$$

The image is the affine variety defined by:

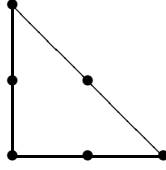
$$(3) \quad v_4^2 + v_2^3 - v_1 v_3 - v_1^{2n+1} v_2 = 0.$$

The associated quiver does not seem to be known in the literature. There is a natural \mathbb{C}^* -action on Y defined as follows:

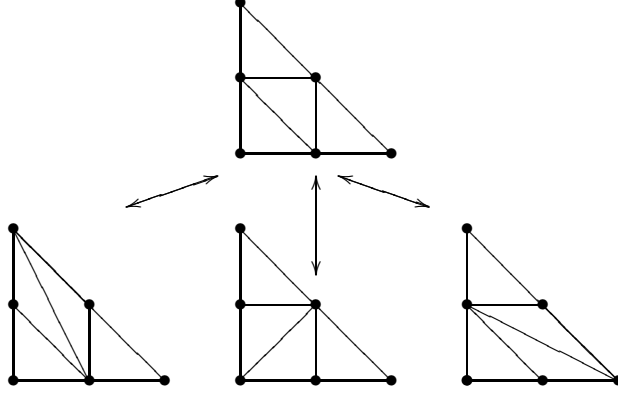
$$\begin{aligned} t \cdot (x, y_1, y_2) &= (t^{1-2n}x, t^{6n+1}y_1, t^2y_2), \\ t \cdot (w, z_1, z_2) &= (t^{2n-1}w, t^4z_1, t^{2n+1}z_2), \end{aligned}$$

where $t \in \mathbb{C}^*$. Using this action one can define and compute the local Gromov-Witten invariants as in [36].

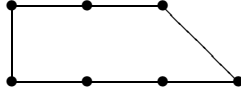
2.2. The case of a string of rational curves as exceptional sets. The case of crepant resolutions of a toric Calabi-Yau 3-fold whose exceptional set is a string of rational curves has been studied by Nagao [27]. There are two cases for the corresponding lattice polygons: Case 1. The triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 2)$:



It has four subdivisions related by flops:



Case 2. The trapezoid with vertices $(0, 0)$, $(N_0, 0)$, $(N_1, 1)$ and $(0, 1)$, where $N_0 \geq N_1 > 0$ are positive integers. For example:

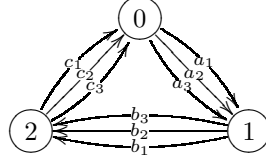


Again there are many different ways to subdivide it into some triangles of area $1/2$.

In both cases, one can use localization to define and compute their local Gromov-Witten invariants, and express the results in terms of the topological vertex [1, 24]. For the second case, see [19].

2.3. The case of surfaces as exceptional sets. When the exceptional set of $\pi : Y \rightarrow X$ is an algebraic surface, it can only be a del Pezzo surface [30], i.e. \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at $k < 9$ points. For the associated quivers and their

superpotentials, see e.g. [2, 4, 13, 18] and the references therein. For example, for $Y = \mathcal{O}_{\mathbb{P}^2}(-3)$, the associated quiver is

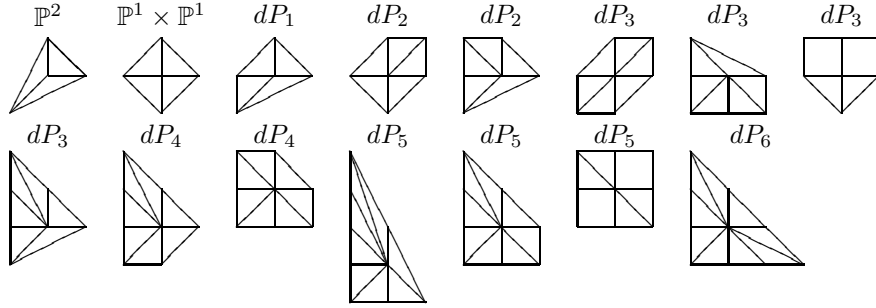


with superpotential function

$$W = a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1.$$

By choosing different strongly exceptional collections, it is possible to arrive at different quivers with superpotentials. For example, in the del Pezzo 3 case, there are four different possibilities (see e.g. [13]).

There are many toric examples (see e.g. [9], Fig. 1):



where dP_n means \mathbb{P}^2 blown up at n points. The local Gromov-Witten invariants of these toric examples have been computed in [37]. For some nontoric cases, see [11, 22].

2.4. Crepant resolutions of 3-dimensional Gorenstein singularities. Recall there is a classification of finite subgroups $\Gamma \subset SL(3, \mathbb{C})$ (cf. [26]). Let $X = \mathbb{C}^3/\Gamma$. By a case by case analysis, one can obtain a crepant resolution $\widehat{\mathbb{C}^3/\Gamma}$ of \mathbb{C}^3/Γ (see Roan [31] and the references given there). Many examples are toric Calabi-Yau 3-fold, so one can use localization to define and compute their local Gromov-Witten invariants, and express the results in terms of the topological vertex [1, 24].

Example 1. Let Γ be generated by

$$\begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix}.$$

Then one can take $\widehat{\mathbb{C}^3/\Gamma} \cong \kappa_{\mathbb{P}^2}$. It can be covered by three local coordinate patches $(u_i, v_i, w_i) \in \mathbb{C}^3$, $i = 1, 2, 3$. Let (z_1, z_2, z_3) be linear coordinates on \mathbb{C}^3 . We take

$$\begin{aligned} u_1 &= \frac{z_2}{z_1}, & v_1 &= \frac{z_3}{z_1}, & w_1 &= z_1^3, \\ u_2 &= \frac{z_1}{z_2}, & v_2 &= \frac{z_3}{z_2}, & w_2 &= z_2^3, \\ u_3 &= \frac{z_1}{z_3}, & v_3 &= \frac{z_2}{z_3}, & w_3 &= z_3^3. \end{aligned}$$

From these we get:

$$u_2 = \frac{1}{u_1}, \quad v_2 = \frac{v_1}{u_1}, \quad w_2 = u_1^3 w_1,$$

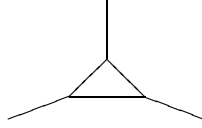
and so on. Let $(\mathbb{C}^*)^3$ act on \mathbb{C}^3 by

$$(t_1, t_2, t_3) \cdot (z_1, z_2, z_3) = (t_1 z_1, t_2 z_2, t_3 z_3).$$

This induces the following action on the $\kappa_{\mathbb{P}^2}$ given in the above local coordinate patches by

$$\begin{array}{lll} u_1 \mapsto \frac{t_2}{t_1} \cdot u_1, & v_1 \mapsto \frac{t_3}{t_1} \cdot v_1, & w_1 \mapsto t_1^3 \cdot w_1, \\ u_2 \mapsto \frac{t_1}{t_2} \cdot u_2, & v_2 \mapsto \frac{t_3}{t_2} \cdot v_2, & w_2 \mapsto t_2^3 \cdot w_2, \\ u_3 \mapsto \frac{t_1}{t_3} \cdot u_3, & v_3 \mapsto \frac{t_2}{t_3} \cdot v_3, & w_3 \mapsto t_3^3 \cdot w_3. \end{array}$$

It is clear that that $\{(u_1, v_1, w_1) \in \mathbb{C}^3 \mid u_1 v_1 w_1 \neq 0\}$ is a dense open orbit of the torus action. Each coordinate patch has a unique fixed point $(u_i, v_i, w_i) = (0, 0, 0)$. Denote them by p_i . The weight decomposition at these points are $(-\alpha + \alpha_2) \oplus (-\alpha_1 + \alpha_3) \oplus (3\alpha_1)$, $(\alpha_1 - \alpha_2) \oplus (-\alpha_2 + \alpha_3) \oplus (3\alpha_2)$, $(\alpha_1 - \alpha_3) \oplus (\alpha_2 - \alpha_3) \oplus (3\alpha_3)$ respectively. The information about the fixed points and the weight decompositions at the fixed points of the torus action can be encoded in the GKM graph [15]:



The toric information can also be encoded in the following lattice polygon:



This coincides with one of the cases we have seen in the previous subsection.

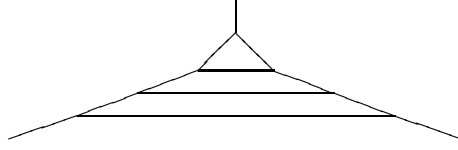
Example 2. In general, for an odd positive integer $2n + 1$, let Γ be generated by the matrix

$$\begin{pmatrix} e^{2\pi i/(2n+1)} & 0 & 0 \\ 0 & e^{2\pi i/(2n+1)} & 0 \\ 0 & 0 & e^{-4\pi i/(2n+1)} \end{pmatrix},$$

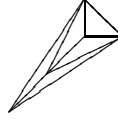
one can explicitly write down a crepant resolution using $2n + 1$ coordinate patches. For example, when $n = 3$, one can take local coordinates:

$$\begin{array}{ll} \left(\frac{z_1}{z_3}, \frac{z_2}{z_3}, z_3^7\right) & \left(\frac{z_1}{z_2}, \frac{z_3}{z_2}, \frac{z_2^3}{z_3^2}\right) \\ \left(\frac{z_2}{z_1}, \frac{z_3^2}{z_1^3}, \frac{z_1^5}{z_3}\right) & \left(\frac{z_1}{z_2}, \frac{z_3^2}{z_2^3}, \frac{z_2^5}{z_3}\right) \\ \left(\frac{z_2}{z_1}, \frac{z_3}{z_1^5}, z_1^7\right) & \left(\frac{z_1}{z_2}, \frac{z_3}{z_2^5}, z_2^7\right). \end{array}$$

The corresponding GKM graph looks like:



The toric information can also be encoded in the lattice polygon with vertices $(1, 0)$, $(0, 1)$, $(-n, -n)$. The crepant resolution corresponding to the subdivision given by adding vertices at $(-k, -k)$, $k = 0, 1, \dots, n-1$. See e.g. the case of $n = 2$:



2.4.1. Three-dimensional McKay correspondence. Another method to obtain the crepant resolution is via Nakamura's Γ -Hilbert schemes [7]. This realizes a crepant resolution of \mathbb{C}^3/Γ as the moduli space of Γ -cluster of points in \mathbb{C}^3 . As a consequence, Bridgeland-King-Reid [7] proved by Fourier-Mukai transform an equivalence of derived categories:

$$(4) \quad D^b(\widehat{\mathbb{C}^3/\Gamma}) \cong D_\Gamma^b(\mathbb{C}^3),$$

where $D^b(\widehat{\mathbb{C}^3/\Gamma})$ denotes the derived category of bounded complexes of coherent sheaves on $\widehat{\mathbb{C}^3/\Gamma}$, and $D_\Gamma^b(\mathbb{C}^3)$ denotes the derived category of bounded complexes of Γ -equivariant coherent sheaves on \mathbb{C}^3 . This is a generalization of the McKay correspondence in two dimensions [16, 21]. Define a noncommutative algebra

$$R_\Gamma = \mathbb{C}[z_1, z_2, z_3] \ltimes \mathbb{C}\Gamma,$$

where $\mathbb{C}[z_1, z_2, z_3]$ is the algebra of polynomials on \mathbb{C}^3 , $\mathbb{C}\Gamma$ is the group ring of Γ , and G_Γ is their twisted product, in particular,

$$g \cdot p(z_1, z_2, z_3) = g(p(z_1, z_2, z_3)) \cdot g,$$

where $g \in \Gamma$, $p(z_1, z_2, z_3)$, $g(p(z_1, z_2, z_3))$ is the action of g on $p(z_1, z_2, z_3)$. Let $R_\Gamma - \text{mod}$ be the category of finitely generated R_Γ -module and let $D^b(R_\Gamma - \text{mod})$ be its derived category. Then we have

$$(5) \quad D_\Gamma^b(\mathbb{C}^3) \cong D^b(R_\Gamma).$$

The noncommutative algebra R_Γ is an example of noncommutative resolution [35]. Note the center of R_Γ is $\mathbb{C}[z_1, z_2, z_3]^\Gamma$, the coordinate ring of \mathbb{C}^3/Γ , and so we have the natural inclusion $\mathbb{C}[z_1, z_2, z_3]^\Gamma \hookrightarrow R_\Gamma$.

2.4.2. Quivers with superpotentials. Given an R_Γ -module M , we can regard it first as a Γ -module and consider a decomposition:

$$M = \bigoplus_{\chi \in \hat{\Gamma}} M_\chi,$$

where $\hat{\Gamma}$ is the set of irreducible representations, and M_χ is a direct sum of irreducible representations of characters χ . Then we consider M as an $\mathbb{C}[z_1, z_2, z_3]$ -module and consider the maps of multiplications by z_1 , z_2 and z_3 . For simplicity we first assume G is abelian, and so all irreducible representations are 1-dimensional.

Furthermore, assume $\mathbb{C}z_i$ is an irreducible representation χ_{z_i} of Γ . Then multiplication by z_i induces a map

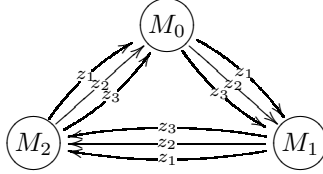
$$z_i : M_\chi \rightarrow M_{\chi \otimes \chi_{z_i}}.$$

This inspires one to consider the following quiver (directed graph): Assign a vertex to each χ , and assign an arrow from χ to $\chi \otimes \chi_{z_i}$ for each $i = 1, 2, 3$. Because multiplication by z_i commutes with multiplication by z_j when $i \neq j$, some relations should be imposed for the quiver, as is clear from the following example.

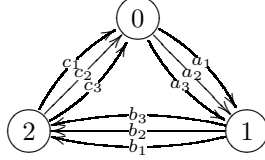
Example 3. Let $\Gamma = \mathbb{Z}_3$ and let ω a generator. There are three irreducible representations χ_0, χ_1 and χ_2 : $\chi_k(\omega) = e^{2k\pi i/3}$. Suppose that

$$(6) \quad \omega \cdot z_k = e^{2\pi i/3} \cdot z_k \omega.$$

The decomposition of the R_Γ -module is described by the following diagram:



This leads to the quiver we have seen in the $\mathcal{O}(-\mathbb{P}^2(-3))$ case:



but now a_1, a_2, a_3 correspond to multiplication by z_1 , b_1, b_2, b_3 correspond to multiplication by z_2 , and c_1, c_2, c_3 correspond to multiplication by z_3 . Therefore, we should impose the following relations:

$$(7) \quad a_i b_j = a_j b_i, \quad b_i c_j = b_j c_i, \quad c_i a_j = c_j a_i,$$

whenever $i \neq j$. These 27 relations can be encoded in the superpotential function:

$$W = a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1.$$

The relations are obtained by taking $\frac{\partial}{\partial a_i}$, $\frac{\partial}{\partial b_i}$ and $\frac{\partial}{\partial c_i}$ for $i = 1, 2, 3$.

For the construction of the quiver with potential (Q^Γ, W_Γ) associated to a general $\Gamma \subset GL(3, \mathbb{C})$, see Ginzburg [14, §4.4]. Its path algebra is Morita equivalent to R_Γ .

3. GW/NC DT CORRESPONDENCE

In last section, we have seen many examples that to a crepant resolution $\pi : Y \rightarrow X$ of affine Calabi-Yau 3-fold with isolated singularity, one can associate a quiver Q with superpotential function W , such that

$$D^b(Y) \cong D^b(Q, W).$$

We have also seen that many examples of the crepant resolutions are toric, and therefore one can use localization to define and compute their local Gromov-Witten invariants. Each of these quivers have a special vertex v_0 , corresponding to \mathcal{O}_Y in the strong exceptional collection that gives rise to Q . Following Nagao-Nakajima

[28] and Nagao [27], we will consider the framed quiver \tilde{Q} by adding a vertex v_∞ and an arrow from v_∞ to v_0 , with the same superpotential function W .

3.1. Moduli spaces of quiver representations. Given a quiver $Q = (Q_0, Q_1)$, where Q_0 denotes the set of vertices, and Q_1 the set of arrows. There are two maps $h, t : Q_1 \rightarrow Q_0$ which specify for each arrow its head and tail.

Let $\{W_v\}_{v \in Q_0}$ a collection of complex vector spaces, one for each vertex of the quiver. Its dimension vector of a representation is the vector $\alpha = (\dim W_v)_{v \in Q_0}$. A representation of Q with dimension vector α is a collection of linear maps $\{\phi_a : W_{ta} \rightarrow W_{ha}\}_{a \in Q_1}$. The dimension vector of a representation is the vector $\alpha = (\dim W_v)_{v \in Q_0}$. They form a vector space

$$\mathcal{R}(Q; \alpha) := \bigoplus_{a \in Q_1} \text{Hom}(W_{ta}, W_{ha}),$$

called the representation space of Q with dimension vector α . There is a natural action of the group

$$GL(\alpha) := \prod_{v \in Q_0} GL(W_v)$$

given by $(g \cdot \phi)_a = g_{ha} \phi_a g_{ta}^{-1}$. To define the moduli spaces of quiver representations, one has to consider the stability conditions. Given a collector of real numbers $\theta = (\theta_v)_{v \in Q_0}$, it gives a stability condition if

$$(8) \quad \sum_{v \in V_0} \theta_v \alpha_v = 0.$$

Given a stability condition θ , a Q -representation (U, ϕ) is θ -semistable (stable) if

$$(9) \quad \sum_{v \in Q_0} \theta_v \dim U_v \geq (>) 0$$

for every subrepresentation. Denote by $\mathcal{R}_{\theta-ss}(Q; \alpha) \subset \mathcal{R}(Q; \alpha)$ the set θ -semistable Q -representations of dimension vector α , and $\mathcal{M}_\theta(Q; \alpha) := \mathcal{R}_{\theta-ss}(Q; \alpha) / GL(\alpha)$ the moduli space. If some relations coming from a potential function Φ are imposed on Q , we denote $\mathcal{R}(Q, \Phi; \alpha) \subset \mathcal{R}(Q; \alpha)$ the subset of representations satisfying these relations, and by $\mathcal{M}_\theta(Q, \Phi; \alpha)$ the corresponding moduli space. We conjecture that for suitable stability conditions and dimension vectors, the moduli spaces give some crepant resolutions.

3.2. Noncommutative Donaldson-Thomas invariants. Now let (Q, W) be the quiver with potential function associated with a crepant resolution $\pi : Y \rightarrow X$ as in §2, let (\tilde{Q}, W) be the corresponding framed quiver. We will consider dimension vectors $\tilde{\alpha}$ of \tilde{Q} such that $\alpha_\infty = 1$. For generic $\theta : Q_0 \rightarrow \mathbb{R}$, extend it to $\tilde{\theta} : Q_0 \cup \{\infty\} \rightarrow \mathbb{R}$ by $\tilde{\theta}_v = \theta_v$ for $v \in Q_0$ and

$$(10) \quad \theta_\infty = - \sum_{v \in Q_0} \theta_v \alpha_v.$$

By the same argument as in Segal [33] and Szendroi [34], $\mathcal{M}_{\tilde{\theta}}(\tilde{Q}, W; \tilde{\alpha})$ has a symmetric perfect obstruction theory, and one can use Behrend's constructible function [5] to define its Donaldson-Thomas type invariants. We will refer to these invariants as noncommutative Donaldson-Thomas (NCDT) invariants. Denote the corresponding partition function by $Z_{\tilde{\theta}}(\tilde{Q}, W)$.

As in [28, 27], we consider all stability conditions. We conjecture that they have some chamber structures, related to the infinite root system associated to Q by the Kac Theorem [20]. Recall one can associate a Cartan matrix to the quiver Q , and use the Weyl reflections to define a root system. There is an indecomposable representation of Q^Γ of dimension vector α if and only if α is a positive root. Furthermore, if α is a positive real root, then there is a unique representation of Q of dimension vector α ; if α is a positive imaginary root, then such representations form a variety of dimension (α, α) . The relations imposed by the potential function W will eliminate some positive roots because the corresponding indecomposable representations do not satisfy the relations. The remaining positive roots will then give rise to hypersurfaces that divide the space of θ 's into chambers.

3.3. GW/NCDT correspondence conjecture. Inspired by Crepant Resolution Conjecture [32, 8, 10], GW/DT correspondence conjecture [25], and recent work of Nagao and Nakajima [28, 27], we make the following

Conjecture 1. *Suppose that (Q, W) is a quiver with superpotential function associated to a crepant resolution $\pi : Y \rightarrow X$, where X is an affine Calabi-Yau 3-fold with an isolated singularity. Let (\tilde{Q}, W) be the framed quiver obtained from (Q, W) . Then the generating series $Z^{GW}(Y)$ of local Gromov-Witten invariants of Y can be identified with the generating series $Z_\theta^{NCDT}(\tilde{Q}, W)$ of NCDT invariants of (\tilde{Q}, W) for suitable stability parameters θ in a chamber C_{GW} .*

We will refer to this conjecture as the noncommutative crepant resolution conjecture or GW/NCDT correspondence conjecture. In joint work with Weiping Li in progress, we are verifying our conjecture in the case of $\pi : \widehat{\mathbb{C}^3/\mathbb{Z}_3} \rightarrow \mathbb{C}^3/\mathbb{Z}_3$.

We conjecture that there is a chamber C_0 such that $Z_\theta(Q, W) = 1$ for $\theta \in C_0$, and there is a sequence of wall-crossings to C_{GW} such that each wall-crossing changes the partition function by multiplying a factor so that $Z_\theta(\tilde{Q}, W)$ becomes an infinite product for $\theta \in C_{GW}$. We expect this corresponds to the infinite product structure of $Z^{GW}(Y)$ predicted by the Gopakumar-Vafa integrality [17]. This may shed some lights on the Gopakumar-Vafa invariants by relating them to infinite root systems.

We speculate that our conjecture follows from more general results, such as the PT type invariants associated to two different t -structures of a Calabi-Yau category can be identified after suitable change of variables, and the relationship between GW invariants, DT invariants and PT invariants of toric Y .

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